Lecture 1: Linear Algebra

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Contents

1.1 Vectors and Matrices

You probably know very well what a vector and a matrix are.

Definition 1.1.1 (Scalars, vectors, and matrices) A "scalar", for the purpose of this course, is either a real ($\mathbb{R}$) or a complex ($\mathbb{C}$) number. When I want to refer to a scalar which can be both a real or a complex number, I use the notation $\mathbb{F}$. In other words, $\mathbb{F}$ means either $\mathbb{R}$ or $\mathbb{C}$. A "vector", again for the purpose of this course, is an ordered set of numbers, depicted as a column:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n = \mathbb{F}^{n\times 1}$$

Almost always, our vectors are column vectors. But occasionally, we need row vectors as well, which we may show using the transpose ^{T} notation:

$$\mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \in \mathbb{F}^{1\times n}$$

And finally a "matrix" is a rectangular ordered set of numbers,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{F}^{m\times n}$$

As you have noticed, throughout this course, we use bold-faced small letters for vectors and bold-faced capital letters for matrices. Also, notice our notation for the spaces of real/complex vectors and matrices. □

Vectors and matrices of the same size can be added together, and both vectors and matrices can be multiplied by a scalar, not super interesting. What is more interesting and, as you will see, essentially the basis of linear algebra, is matrix multiplication. You probably have seen the basic definition.

Definition 1.1.2 (Matrix multiplication) For two matrices $\mathbf{A} \in \mathbb{F}^{m\times n}$ and $\mathbf{B} \in \mathbb{F}^{r\times p}$, their product $\mathbf{C} = \mathbf{AB}$ is only defined if $n = r$, in which case the entries of $\mathbf{C}$ are defined as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

□

The above definition, however, gives little intuition about what a matrix multiplication really does. To see this, we need to notice two facts.

Theorem 1.1.3 (Break-down of matrix multiplication) Let $\mathbf{a}_i$ and $\mathbf{b}_i$ denote the i'th column of $\mathbf{A}$ and $\mathbf{B}$, respectively:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$$

Then

(i) The matrix-matrix multiplication $\mathbf{AB}$ applies to each column of $\mathbf{B}$ separately, that is

$$\mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \cdots & \mathbf{A}\mathbf{b}_p \end{bmatrix} \tag{1.1}$$

In other words, the i'th column of $\mathbf{AB}$ is $\mathbf{A}$ times the i'th column of $\mathbf{B}$.

(ii) Each matrix-vector multiplication $\mathbf{A}\mathbf{b}_i$ is a weighted sum of the columns of $\mathbf{A}$, that is

$$\mathbf{A}\mathbf{b}_i = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{bmatrix}\begin{bmatrix}b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni}\end{bmatrix} = b_{1i}\mathbf{a}_1 + b_{2i}\mathbf{a}_2 + \cdots + b_{ni}\mathbf{a}_n \tag{1.2}$$

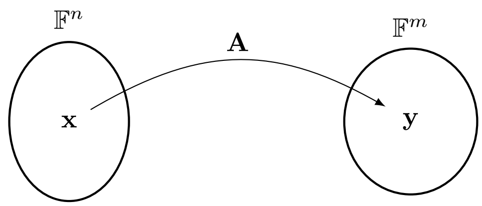
□

You can easily show both of these properties using Definition 1.1.2, but they are going to be super useful in understanding linear algebra as it really is.

You might be wondering why I put so much emphasis on these simple properties? One reason is the intuition that they give you on matrix multiplication. But it's more than that. The real reason is the important role that matrix-vector multiplication plays in linear algebra. Notice that for a matrix $\mathbf{A} \in \mathbb{F}^{m\times n}$ and a vector $\mathbf{x} \in \mathbb{F}^n$, their product

$$\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{F}^m$$

is also a vector. In other words, a matrix is more than a rectangular array of numbers! It defines a function that maps vectors to vectors.



One of the most interesting and intuitive examples of matrices as maps comes from rotation matrices:

1.2 Rotation Matrices and Special Orthogonal Groups

You probably remember that in 2 dimensions (plane), the matrix

$$\mathbf{R}_\theta = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

rotates any two dimensional vector $\mathbf{x} = \begin{bmatrix}x_1 \\ x_2\end{bmatrix}$ around the origin by the angle θ counter-clockwise. This matrix has two critical properties:

$\mathcal{P}_1$: $\det(\mathbf{R}_\theta) = 1$

$\mathcal{P}_2$: $\mathbf{R}_\theta^{-1} = \mathbf{R}_\theta^T$

Before discussing these properties further, it may be helpful to give a refresher on determinants and inverses.

Definition 1.2.1 (Determinant) Consider any n-by-n matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

- If $n = 1$ (a is a scalar), then its determinant equals itself

$$\det(a) = a$$

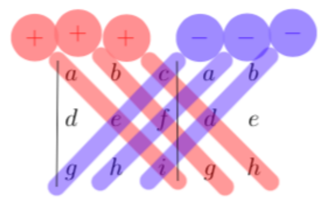
- If $n = 2$,

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\det(\mathbf{A}) = ad - bc$$

- If $n = 3$,

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
$$\det(\mathbf{A}) = aei + bfg + cdh - ceg - bdi - afh$$

Which is much easier to remember and calculate using the following image:



- If $n > 1$ (including $n = 2$ and 3 above, but really used for $n \geq 4$), then the determinant of $\mathbf{A}$ is defined based on an expansion over any arbitrary row or column. For instance, choose row 1. Then

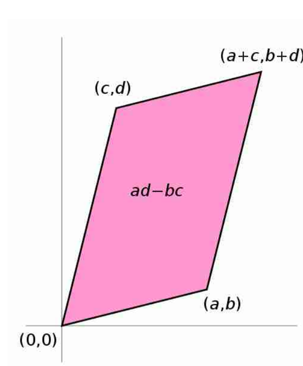
$$\det(\mathbf{A}) = |\mathbf{A}| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(\mathbf{A}_{-(1,j)})$$

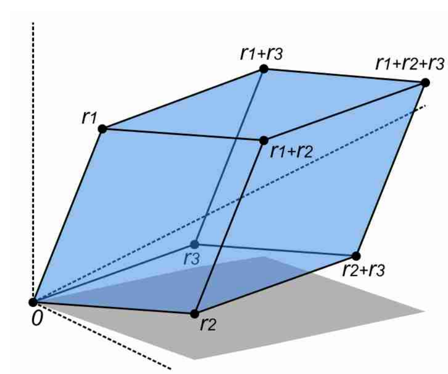
where $\mathbf{A}_{-(1,j)}$ is an $n-1$ by $n-1$ matrix obtained from $\mathbf{A}$ by removing its 1st row and j'th column.

□

The determinant also has a nice geometrical interpretation,

$$\begin{aligned} \text{when } n = 2: \quad & |\det(\mathbf{A})| = \text{ area of the parallelogram formed by the columns (or rows) of } \mathbf{A} \\ \text{when } n = 3: \quad & |\det(\mathbf{A})| = \text{ volume of the parallelepiped formed by the columns (or rows) of } \mathbf{A} \end{aligned}$$





as well as the important properties

- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$;
- $\det(\mathbf{A}^T) = \det(\mathbf{A})$;
- Determinant of any upper- or lower-triangular (including diagonal) matrix is equal to the product of its diagonal entries.

If a matrix has $\det(\mathbf{A}) \neq 0$, then it is *invertible*, meaning, there exists another matrix denoted by $\mathbf{A}^{-1}$ such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix (the equivalent of number 1 for matrix multiplication). If $\det(\mathbf{A}) = 0$, the matrix is called *singular* and no such inverse matrix exists.

Exercise 1.2.2 (Determinant of inverse) Show that $\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$. □

Now we can pick up the discussion on the determinants and inverses of $\mathbf{R}_\theta$:

- $\det(\mathbf{R}_\theta) = 1$ because

$$\det(\mathbf{R}_\theta) = \cos^2\theta + \sin^2\theta = 1$$

- $\mathbf{R}_\theta^{-1} = \mathbf{R}_\theta^T$ because

$$\mathbf{R}_\theta\mathbf{R}_\theta^T = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

  and similarly $\mathbf{R}_\theta^T\mathbf{R}_\theta = \mathbf{I}$.

These two properties are in fact unique to rotation matrices:

Exercise 1.2.3 (2D rotation matrices) Prove that any 2x2 matrix with properties $\mathcal{P}_1$ and $\mathcal{P}_2$ is necessary a rotation matrix $\mathbf{R}_\theta$ for some $\theta \in (-\pi, \pi]$. □

Therefore, a rotation matrix can be defined as any matrix with properties $\mathcal{P}_1$ and $\mathcal{P}_2$. Why this may not sound very useful in 2D, it becomes quite non-trivial in 3D. A couple of definitions are in order:

Definition 1.2.4 (Orthogonal matrix) Any real square matrix (of any size) that satisfies the property $\mathcal{P}_2$ is called an orthogonal matrix. □

Definition 1.2.5 (Special orthogonal groups) In any dimension n, the set of all real square matrices satisfying properties $\mathcal{P}_1$ and $\mathcal{P}_2$ is called the special orthogonal group of $\mathbb{R}^{n\times n}$ and denoted by $SO(n)$. □

Let's first take a look at Definition 1.2.4. Recall that for any two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, their inner product is the scalar (number)

$$\mathbf{v}^T\mathbf{w} = \mathbf{w}^T\mathbf{v} = \sum_{i=1}^n v_i w_i$$

which becomes the square of vector Euclidean norm when $\mathbf{v} = \mathbf{w}$,

$$\|\mathbf{v}\|^2 = \mathbf{v}^T\mathbf{v} = \sum_{i=1}^n v_i^2$$

and otherwise satisfies

$$\mathbf{v}^T\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Now assume some matrix $\mathbf{V} \in \mathbb{R}^{n\times n}$ is orthogonal and let $\mathbf{v}_i$ be its i'th column,

$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}$$

Then

$$\mathbf{V}^T\mathbf{V} = \begin{bmatrix}\mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T\end{bmatrix} \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\|\mathbf{v}_1\|^2 & \mathbf{v}_1^T\mathbf{v}_2 & \cdots & \mathbf{v}_1^T\mathbf{v}_n \\ \mathbf{v}_2^T\mathbf{v}_1 & \|\mathbf{v}_2\|^2 & \cdots & \mathbf{v}_2^T\mathbf{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_n^T\mathbf{v}_1 & \mathbf{v}_n^T\mathbf{v}_2 & \cdots & \|\mathbf{v}_n\|^2\end{bmatrix}$$

So $\mathbf{V}$ being an orthogonal matrix means that

$$\begin{cases}\|\mathbf{v}_i\| = 1 & \forall i \\ \mathbf{v}_i^T\mathbf{v}_j = 0 & \forall i \neq j\end{cases}$$

In other words, orthogonal matrices are matrices whose columns (and rows, proven similarly) all have unit norm (length) and are pairwise orthogonal (hence the name "orthogonal matrix"). This is of course very restrictive, and notice that if $\mathbf{V}$ is orthogonal,

$$\begin{aligned}\mathbf{V}^T\mathbf{V} = \mathbf{I} &\Rightarrow \det(\mathbf{V})^2 = 1 \\ &\Rightarrow \det(\mathbf{V}) = \pm 1\end{aligned}$$

In other words, property $\mathcal{P}_2$ *almost* implies property $\mathcal{P}_1$ automatically, except for a sign.

Exercise 1.2.6 (Reflections) Show that any orthogonal matrix in 2D that is not a rotation is the combination of a rotation and a reflection (where the -1 comes from). □

Almost all of what we said generalizes beautifully to 3D. In 3D, unlike 2D, there are multiple ways to geometrically define/characterize a rotation (cf. [1, §2.5]). The most intuitive is the axis-angle representation, where a rotation is defined by a unit vector $\mathbf{k}$ (the axis) and a scalar θ (the angle). Then, it can be shown that any matrix $\mathbf{R} = [r_{ij}] \in SO(3)$ defines a rotation around

$$\mathbf{k} = \frac{1}{2\sin\theta}\begin{bmatrix}r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12}\end{bmatrix}$$

with the angle

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

Note that if $\theta = 0$, the corresponding rotation matrix is $\mathbf{R} = \mathbf{I}$ and the axis of rotation is not defined.

Exercise 1.2.7 (3D rotation of π radians) What happens with $\theta = \pi$? How can you find the axis of rotation in this case? You can use [1, Eq. (2.44)]. □

1.3 Matrix Exponential and Exponential Coordinates of Rotation

Recall that for any scalar x, its exponential is defined through the infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

Likewise, for any square matrix $\mathbf{A}$, its matrix exponential is defined through the infinite series

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

In both cases, these series are guaranteed to converge for all scalars and matrices. The matrix exponential satisfies the following properties, partly analogous to scalar exponentials and partly different from them:

(i) $e^{\mathbf{0}} = \mathbf{I}$

(ii) $(e^{\mathbf{A}})^T = e^{\mathbf{A}^T}$

(iii) $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}} e^{\mathbf{B}}$ if $\mathbf{A}$ and $\mathbf{B}$ commute (i.e., $\mathbf{AB} = \mathbf{BA}$)

(iv) $\det(e^{\mathbf{A}}) = e^{\mathrm{tr}(\mathbf{A})}$ where $\mathrm{tr}(\mathbf{A})$ is the trace of $\mathbf{A}$

(v) $\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t}$

(vi) $e^{\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}} = \mathbf{Q}^{-1} e^{\mathbf{A}} \mathbf{Q}$

The matrix exponential usually comes up in the solutions of linear time-invariance systems, but it also has other interesting properties, such as the following:

Theorem 1.3.1 (Rotations and matrix exponential) Consider any $\mathbf{R} \in SO(3)$ and let $(\mathbf{k}, \theta)$ denote its axis-angle representation. Define the skew-symmetric matrix

$$S(\mathbf{k}) = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

Then

$$\mathbf{R} = e^{S(\mathbf{k})\theta}$$

□

Note that $S(\mathbf{k})\theta = S(\theta\mathbf{k})$, giving a compact three-parameter representation $\theta\mathbf{k}$ of any rotation which is called the *exponential coordinates* of that rotation.

1.4 Vector Spaces

Linear algebra, arguably, is the study of matrices as maps. We planted the seed of this view of matrices at the end of Section 1.1 but it should be much clearer now after seeing the rotation matrices. The sets $\mathbb{F}^n$ or $\mathbb{F}^m$ (the domain and co-domain of the map defined by a matrix $\mathbf{A} \in \mathbb{F}^{m\times n}$, not surprisingly, are called "vector spaces" – they are spaces of vectors after all! So more formally, a matrix defines a function that maps one vector space onto another. But there is more to vector spaces – not any set of vectors is a vector space.

Notice that the vector space $\mathbb{F}^m$ has two, rather trivial, properties:

(i) For any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{F}^m$, their sum $\mathbf{x} + \mathbf{y}$ is also in $\mathbb{F}^m$;

(ii) For any $\mathbf{x}$ in $\mathbb{F}^m$ and any scalar $\alpha \in \mathbb{F}$, the scalar product $\alpha\mathbf{x}$ is also in $\mathbb{F}^m$.

But importantly, there are more sets that are not of the form $\mathbb{F}^m$ but still have these properties. For example, consider the set of vectors

$$V = \left\{ \begin{bmatrix} 3a \\ 4a \end{bmatrix} \mid a \in \mathbb{F} \right\}$$

Clearly, for any $\begin{bmatrix} 3a \\ 4a \end{bmatrix}$ and $\begin{bmatrix} 3b \\ 4b \end{bmatrix}$ in V, so is $\begin{bmatrix} 3a \\ 4a \end{bmatrix} + \begin{bmatrix} 3b \\ 4b \end{bmatrix} = \begin{bmatrix} 3(a+b) \\ 4(a+b) \end{bmatrix}$. Also, for any $\alpha \in \mathbb{F}$, $\alpha \begin{bmatrix} 3a \\ 4a \end{bmatrix} = \begin{bmatrix} 3(\alpha a) \\ 4(\alpha a) \end{bmatrix}$ is clearly in V as well. Any such set is called a vector space.

Definition 1.4.1 (Vector space) A set $V \subseteq \mathbb{F}^n$ of vectors is a "vector space" if

(P1) For any $\mathbf{x}$ and $\mathbf{y}$ in V, their sum $\mathbf{x} + \mathbf{y}$ is also in V;

(P2) For any $\mathbf{x}$ in V and any scalar $\alpha \in \mathbb{F}$, the scalar product $\alpha\mathbf{x}$ is also in V;

or, equivalently,

(P12) For any $\mathbf{x}$ and $\mathbf{y}$ in V and any two scalars $\alpha, \beta \in \mathbb{F}$, the vector $\alpha\mathbf{x} + \beta\mathbf{y}$ is also in V.

□

At this point, the notion of a vector space may be pretty abstract. What are examples of vector spaces other than $\mathbb{F}^m$ and the V I gave above? To get to the bottom of vector spaces, we need a couple of other concepts: "linear combination", "span", "linear (in)dependence", "rank", and "basis". We go one by one.

Definition 1.4.2 (Linear combination & span) A linear combination of a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is any weighted sum of them, that is, any vector

$$\mathbf{x}_{n+1} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n \tag{1.3}$$

for some scalars (also called "coefficients") $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. The "span" of a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is then the set of all their linear combinations, and is denoted as

$$\langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \rangle = \{\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{F}\}.$$

□

This was hopefully easy, but is important. In particular, convince yourself that

Theorem 1.4.3 (Spans of vectors and vector spaces) The span of any set of vectors is a vector space. □

Now you can see how I made up the $V = \left\{ \begin{bmatrix} 3a \\ 4a \end{bmatrix} \mid a \in \mathbb{F} \right\} = \langle \begin{bmatrix} 3 \\ 4 \end{bmatrix} \rangle$ above. And to prove this theorem, you can basically do what I did above for V in a more general setting. But is the converse also true? That is, is any vector space the span of a set of vectors? The answer is yes. To see how, we need to know about linear (in)dependence of vectors first.

For obvious reasons, the vector $\mathbf{x}_{n+1}$ in Eq. (1.3) is said to be *linearly dependent* on the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ (because you can obtain $\mathbf{x}_{k+1}$ from a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$). Notice that this is not always the case. For example, you cannot find any linear combination of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

that gives you

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and, so, $\mathbf{x}_3$ is *linearly independent* from $\mathbf{x}_1$ and $\mathbf{x}_2$. But

$$\mathbf{x}_4 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

is in fact linearly dependent on $\mathbf{x}_1$ and $\mathbf{x}_2$ (right?).

Here, you might have noticed that not only $\mathbf{x}_3$ cannot be obtained from any linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$, but $\mathbf{x}_2$ cannot be obtained from any linear combination of $\mathbf{x}_1$ and $\mathbf{x}_3$ either, and same for $\mathbf{x}_1$. On the other hand, not only

$$\mathbf{x}_4 = 2\mathbf{x}_1 + 3\mathbf{x}_2$$

but also

$$\mathbf{x}_1 = \frac{1}{2}\mathbf{x}_4 - \frac{3}{2}\mathbf{x}_2$$

and

$$\mathbf{x}_2 = \frac{1}{3}\mathbf{x}_4 - \frac{2}{3}\mathbf{x}_1$$

In other words, (assuming that no coefficients are 0) linear dependence and linear independence are symmetric properties *between* a set of vectors: they are either all linearly dependent on each other, or neither is linearly dependent on the rest. The formal version is:

Definition 1.4.4 (Linear (in)dependence) A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are "linearly dependent" if there exists a set of scalars $\alpha_1, \ldots, \alpha_n$, at least one of which is not equal to 0, such that

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = 0 \tag{1.4}$$

In contrast, if Eq. (1.4) holds only for $\alpha_1 = \cdots = \alpha_n = 0$ (which it clearly always does), then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are called "linearly independent". □

Now, if I give you a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, how can you say if they are linearly independent or not? For example,

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \\ 3 \end{bmatrix} \tag{1.5a}$$

are linearly independent, but

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \\ -3 \end{bmatrix} \tag{1.5b}$$

are not. There are various ways for determining it, but at the most basic level, you can always check directly from the definition:

⋆ *Determining linear independence from definition:* Solve Eq. (1.4) for $\alpha_1, \ldots, \alpha_n$ using elimination of variables (a.k.a. the substitution method). Any of the α_i's that remains free corresponds to a vector $\mathbf{x}_i$ that is linearly dependent on the rest. If no α_i's remain free (all end up zero), then the set of vectors is linearly independent.

In a few cases, determining linear independence becomes quite simple:

(i) For two vectors, if they are linearly dependent, it means that

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 = 0$$

for some α_1 and α_2 that at least one of them, say α_1, is nonzero. This means that

$$\mathbf{x}_2 = -\frac{\alpha_2}{\alpha_1}\mathbf{x}_1$$

or, in words, two vectors are linearly dependent if and only if they are a multiple of each other.

(ii) If you have n, m-dimensional vectors and $n > m$, they are necessarily linearly dependent. So in $\mathbb{F}^2$, you cannot have 3 linearly independent vectors, in $\mathbb{F}^3$, you cannot have 4 linearly independent vectors, and so on. We will see why shortly, in Section 1.4.7.

OK, back to vector spaces and the converse of Theorem 1.4.3. Consider any vector space

$$V = \langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \rangle \subseteq \mathbb{F}^n. \tag{1.6}$$

We say $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ span V. These vectors may or may not be linearly independent. If they are linearly independent, great! If not, we can drop one or more of them so that the remaining ones are linearly independent and still span V. In other words, we are always looking for a *minimal* set of vectors that span a vector space. Such a minimal spanning set is so important that we call it a "basis" for the vector space:

Definition 1.4.5 (Basis & dimension of vector space) Any set of linearly independent vectors that span a vector space are called a "basis" for that vector space. Any vector space has infinitely many bases, but all of them have the same number of elements, which is called the "dimension" of that vector space. □

If you are unsure why or how a vector space can have infinitely many bases, notice this. For any vector space V as in Eq. (1.6), if $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a basis, so is

$$\{\beta\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \tag{1.7a}$$

for any $\beta \in \mathbb{F}$ (right?). This is already infinitely many bases, but there are way more. For example, any

$$\{\beta_1\mathbf{x}_1, \beta_2\mathbf{x}_2, \ldots, \beta_n\mathbf{x}_n\} \tag{1.7b}$$

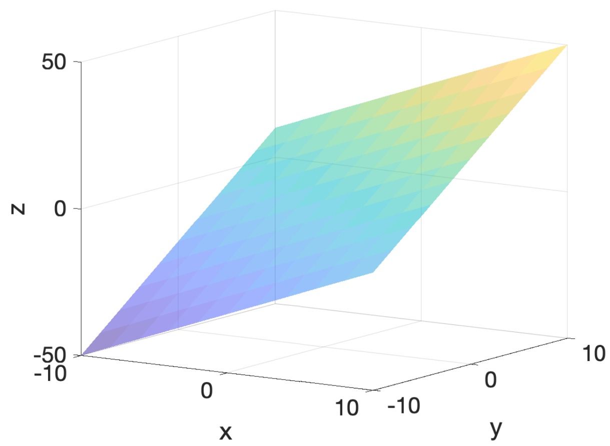
for any $\beta_1, \ldots, \beta_n$ is also a basis. Even more,

$$\{\mathbf{x}_1, \mathbf{x}_2 + \beta\mathbf{x}_1, \ldots, \mathbf{x}_n\} \tag{1.7c}$$

is also a basis, and so on.

Example 1.4.6 (2D vector space in 3D) Consider the following plane in the 3-dimensional space

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid z = 2x + 3y \right\}$$



This is indeed a vector space. To see why, notice that any vector $\mathbf{v} \in V$ satisfies

$$v = \begin{bmatrix} x \\ y \\ 2x + 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

which means

$$V = \langle \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \rangle$$

and so it is a vector space according to Theorem 1.4.3. Since $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ are clearly not multiples of each other, they are linearly independent, and therefore they form a basis for V. Other bases can be formed using any two vectors in V that are linearly independent. For example,

$$x = y = 1 \to \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$x = -y = 1 \to \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

are also linearly independent and span V, and therefore make another basis for V. To emphasize the importance of linear independence, notice that

$$V = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle$$

but neither $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ nor $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are a basis for V. □

Finally, notice that any vector space necessarily contains 0, because 0 is a linear combination of any set of vectors (with 0 coefficients for all the vectors). Therefore, if any set does not contain 0, such as

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid z = 2x + 3y + 1 \right\}$$

we can be sure that it is not a vector space.

1.4.7 Matrix Rank

Assume that we have a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{F}^m$ and place them side by side into a matrix (as columns):

$$\mathbf{A} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

So the matrix $\mathbf{A}$ has k columns and m rows. Notice that this same matrix can also be seen as a stack of m, n-dimensional row vectors

$$\mathbf{A} = \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_m^T \end{bmatrix}$$

where each row vector $\mathbf{y}_i^T$ denotes the i'th row of $\mathbf{A}$ (or, equivalently, the column vector $\mathbf{y}_i$ is the transpose of the i'th row of $\mathbf{A}$). The number of linearly independent rows and columns of this matrix is a fundamental property of it, called the *rank*:

Definition 1.4.8 (Rank) Consider a matrix $\mathbf{A} \in \mathbb{F}^{m \times n}$ as above. The column-rank of $\mathbf{A}$ is the number linearly independent columns of $\mathbf{A}$ (the number of linearly independent $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$), while the row-rank of $\mathbf{A}$ is the number of linearly independent rows of it (the number of linearly independent $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m$).□

The notion of rank essentially translate a property of a set of vectors to a property of a matrix, but is very fundamental to linear algebra. Let's see a couple of examples.

Example 1.4.9 (Rank of matrices) Consider again the example vectors in Eq. (1.5). The first set of vectors were linearly independent (right?). Therefore, when we put them side by side in the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 3 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

it has column rank equal to 3. What about its row rank? The rows of the matrix are

$$\mathbf{y}_1^T = \begin{bmatrix} 2 & -3 & 7 \end{bmatrix}, \qquad \mathbf{y}_2^T = \begin{bmatrix} 0 & 3 & -3 \end{bmatrix}, \qquad \mathbf{y}_3^T = \begin{bmatrix} -1 & 1 & 3 \end{bmatrix}$$

which are also linearly independent (check). So the row rank of the matrix is also 3.

Now, consider the vectors in Eq. (1.5b). They are not linearly independent, because $\mathbf{x}_3 = 2\mathbf{x}_1 - \mathbf{x}_2$. But $\mathbf{x}_1$ and $\mathbf{x}_2$ are indeed linearly independent, because they are not a multiple of each other (remember from last section). So putting them side by side, the matrix

$$\mathbf{B} = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 3 & -3 \\ -1 & 1 & -3 \end{bmatrix}$$

has column rank 2. What about its row rank? The rows of $\mathbf{B}$ are

$$\mathbf{y}_1^T = \begin{bmatrix} 2 & -3 & 7 \end{bmatrix}, \qquad \mathbf{y}_2^T = \begin{bmatrix} 0 & 3 & -3 \end{bmatrix}, \qquad \mathbf{y}_3^T = \begin{bmatrix} -1 & 1 & -3 \end{bmatrix}$$

which are also not linearly independent, because $\mathbf{y}_3 = -\frac{1}{2}\mathbf{y}_1 - \frac{1}{6}\mathbf{y}_2$. And similar to $\mathbf{x}_1$ and $\mathbf{x}_2$, $\mathbf{y}_1$ and $\mathbf{y}_2$ are also linearly independent because they are not a multiple of each other. So, exactly 2 of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are linearly independent, and the row rank of $\mathbf{B}$ is also 2. □

The fact that the column rank and the row rank of both $\mathbf{A}$ and $\mathbf{B}$ were equal was not a coincident. This is always the case!

Theorem 1.4.10 (Rank) For any matrix $\mathbf{A} \in \mathbb{F}^{m\times n}$, its column rank equals its row rank, which is called the rank of the matrix. As a consequence,

$$\text{rank}(\mathbf{A}) < \min\{m, n\}. \tag{1.8}$$

□

Recall that in the previous section (second point after Definition 1.4.4), I told you that if you have n, m-dimensional vectors and $n > m$, they are necessarily linearly dependent. Now you can see why from 1.8. For example, consider the following 5, 3-dimensional vectors stacked side by side into the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -2 & 3 & -3 \\ 3 & 1 & 0 & -2 & 2 \\ -3 & -3 & 3 & 3 & -3 \end{bmatrix}$$

The rank of the matrix can at most be 3, because it has only 3 rows and its row-rank (number of independent rows) cannot be more than the number of rows! In this case, the rank is in fact 3, which means that out of the 5 columns, no more than 3 of them can be simultaneously independent form each other.

Note, also, that column rank = 3 does not mean that any selection of 3 columns are linearly independent. Clearly, the last column is minus the 4th column, so any selection of 3 columns that include both the fourth and last columns cannot be linearly independent. Instead, rank = 3 means that at least one selection of 3 columns are linearly independent. For example, the first, second, and third columns are linearly independent, and so are the first, second, and fourth columns.

The rank of a matrix has also an interesting relationship with the determinant:

Theorem 1.4.11 (Rank and determinant) Consider a matrix $\mathbf{A} \in \mathbb{F}^{m\times n}$.

(i) If $m = n$ (square matrix), then

$$\det(\mathbf{A}) \neq 0 \Leftrightarrow \text{rank}(\mathbf{A}) = n \Leftrightarrow \text{ all columns/rows of } \mathbf{A} \text{ are linearly independent}$$

(ii) In general (square matrix or not),

$$\text{rank}(\mathbf{A}) = \text{ dimension of largest square sub-matrix that is nonsingular}$$

A matrix is called "nonsingular" if its determinant is nonzero, and "singular" otherwise. □

To see how to apply this theorem, let us revisit Example 1.4.9. First,

$$\det(\mathbf{A}) = 36 \neq 0$$

so the rank of $\mathbf{A}$ is 3 and all of its columns/rows are independent. For $\mathbf{B}$, we have

$$\det(\mathbf{B}) = 0$$

and so the rank of $\mathbf{B}$ cannot be 3. But still, its rank may be 2, 1, or even 0 (which is clearly not the case, because the rank of a matrix is only 0 if all of its entries are 0). To see if its rank is 2 or 1, we have to find the largest square sub-matrix that is nonsingular. Easily, you can find many 2×2 nonsingular submatrices, for example

$$\begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 7 \\ -1 & -3 \end{bmatrix}$$

so the rank of $\mathbf{B}$ is 2. To also see an example of a matrix with rank 1, take

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

All columns (or rows) are multiples of each other, and you cannot find any 2×2 nonsingular submatrices. You can, however, find 1×1 nonsingular submatrices (any entry of $\mathbf{C}$), which means $\text{rank}(\mathbf{C}) = 1$.

MATLAB 1.4.12 (Rank & determinant) In MATLAB, use the function `rank()` to obtain the rank of a matrix and `det()` to obtain its determinant! □

1.5 Change of Basis

In the Section 1.4, we learned how to determine if any set of vectors are linearly independent. Now, assume we already have a set of linearly independent vectors that form a basis for a vector space V:

$$V = \langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \rangle$$

In Section 1.4 (right after Definition 1.4.5), I explained why V has infinitely many other bases as well, but did not quite say how to obtain *all* of its bases. If you notice, all of the bases that I constructed in Eq. (1.7) have one property in common: they consist of exactly n vectors, each of which is a linear combination of the "old basis" $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. But clearly, you do not get a basis for V from *any* set of n vectors which are linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. A trivial example is

$$\mathbf{y}_1 = \mathbf{x}_1, \mathbf{y}_2 = \mathbf{x}_1, \ldots, \mathbf{y}_n = \mathbf{x}_1$$

All of $\mathbf{y}_i$'s are linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, but they are all equal, so their span is one dimensional. So when do n linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ form a basis for V?

To get the answer, let us consider a general set of n linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$:

$$\begin{aligned} \mathbf{y}_1 &= p_{11}\mathbf{x}_1 + p_{21}\mathbf{x}_2 + \cdots + p_{n1}\mathbf{x}_n \\ \mathbf{y}_2 &= p_{12}\mathbf{x}_1 + p_{22}\mathbf{x}_2 + \cdots + p_{n2}\mathbf{x}_n \\ &\vdots \\ \mathbf{y}_n &= p_{1n}\mathbf{x}_1 + p_{2n}\mathbf{x}_2 + \cdots + p_{nn}\mathbf{x}_n \end{aligned}$$

Remember that for $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ to form a basis for V, they must have two properties:

(i) be linearly independent

(ii) span V

To see if they are independent, we have to check the rank of the matrix

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_n \end{bmatrix} \tag{1.9}$$

whose columns are $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$. But remember from Eq. (1.2) that

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{X}\mathbf{p}_1 \\ \mathbf{y}_2 &= \mathbf{X}\mathbf{p}_2 \\ &\vdots \\ \mathbf{y}_n &= \mathbf{X}\mathbf{p}_n \end{aligned}$$

where

$$\mathbf{X} = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n\end{bmatrix}$$

and

$$\mathbf{p}_1 = \begin{bmatrix}p_{11} \\ p_{21} \\ \vdots \\ p_{n1}\end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix}p_{12} \\ p_{22} \\ \vdots \\ p_{n2}\end{bmatrix}, \quad \cdots \quad \mathbf{p}_n = \begin{bmatrix}p_{1n} \\ p_{2n} \\ \vdots \\ p_{nn}\end{bmatrix}$$

Plugging these into Eq. (1.9), we get

$$\mathbf{Y} = \begin{bmatrix}\mathbf{X}\mathbf{p}_1 & \mathbf{X}\mathbf{p}_2 & \cdots & \mathbf{X}\mathbf{p}_n\end{bmatrix}$$

But now this has the exact form of Eq. (1.1), so

$$\mathbf{Y} = \mathbf{XP}$$

where $\mathbf{P}$ is an n by n matrix of coefficients with $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ as its columns. So now all we need to know is whether $\mathbf{XP}$ has full column rank (i.e., independent columns). Here it is:

Theorem 1.5.1 (Rank of product) Assume $\mathbf{X} \in \mathbb{F}^{m\times n}$ is full column rank. Then $\mathbf{Y} = \mathbf{XP}$ is also full column rank if and only if $\mathbf{P}$ is nonsingular. □

Notice that if $m = n$, this theorem simply follows from the fact that $\det(\mathbf{Y}) = \det(\mathbf{X})\det(\mathbf{P})$. But in general, whether $m = n$ or $m > n$, this theorem ensures that $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are linearly independent if and only if $\mathbf{P}$ is nonsingular. This ensures the requirement (i) above. Trust me, it also ensures requirement (ii)!

So to summarize, if you have one basis $\mathbf{X}$ for a vector space V,

$$\{\mathbf{Y} = \mathbf{XP} \mid \det(\mathbf{P}) \neq 0\}$$

gives the set of *all* possible bases for V.

Example 1.5.2 (Change of basis) Recall Example 1.4.6 again. There, we showed that

$$\mathbf{X} = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 2 & 3\end{bmatrix}$$

was a basis for V, and so was

$$\mathbf{Y} = \begin{bmatrix}1 & 1 \\ 1 & -1 \\ 5 & -1\end{bmatrix}$$

It is not hard to see that

$$\begin{bmatrix}1 & 1 \\ 1 & -1 \\ 5 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}$$

so $\mathbf{Y}$ can be obtained from $\mathbf{X}$ using $\mathbf{P} = \begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}$. But now you know that any basis of V has the form

$$\begin{bmatrix}1 & 0 \\ 0 & 1 \\ 2 & 3\end{bmatrix}\begin{bmatrix}p_{11} & p_{12} \\ p_{21} & p_{22}\end{bmatrix} = \begin{bmatrix}p_{11} & p_{12} \\ p_{21} & p_{22} \\ 2p_{11} + 3p_{21} & 2p_{12} + 3p_{22}\end{bmatrix}$$

subject to the condition

$$p_{11}p_{22} - p_{12}p_{21} \neq 0.$$

□

Now, imagine that you have an "old" basis $\mathbf{X}$ and have an arbitrary vector $\mathbf{v}$ that is written as a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$:

$$\mathbf{v} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{X}\underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}}_{\boldsymbol{\alpha}}$$

The vector of coefficients $\boldsymbol{\alpha}$ is essentially the "representation" of $\mathbf{v}$ in the basis $\mathbf{X}$. Now, we want to change our basis to a "new" basis $\mathbf{Y} = \mathbf{XP}$. What would be the representation of $\mathbf{v}$ in the new basis? Well, that is easy:

$$\mathbf{v} = \mathbf{X}\boldsymbol{\alpha} = \mathbf{Y}\mathbf{P}^{-1}\boldsymbol{\alpha}$$

so $\mathbf{P}^{-1}\boldsymbol{\alpha}$ gives the new representation of $\mathbf{v}$.

So far, we have talked above change bases from an arbitrary basis $\mathbf{X}$ to another arbitrary basis $\mathbf{Y}$. But in real-world situations, by far the most common example of change of basis is from the so-called "standard basis"

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix}, \quad \cdots \quad \mathbf{e}_n = \begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}$$

to another basis $\mathbf{V}$, or vice versa. In this case, either $\mathbf{X} = \mathbf{I}$, or $\mathbf{Y} = \mathbf{I}$, further simplifying the change of basis. If we are changing from the standard basis to $\mathbf{V}$, then $\mathbf{X} = \mathbf{I}$ and $\mathbf{Y} = \mathbf{V}$, so the matrix that performs that change of basis is $\mathbf{P} = \mathbf{V}$ itself. In contrast, we are changing from a basis $\mathbf{V}$ to the standard basis, we will have $\mathbf{X} = \mathbf{V}$ and $\mathbf{Y} = \mathbf{I}$, so the matrix that performs that change of basis is $\mathbf{P} = \mathbf{V}^{-1}$.

1.5.3 Similarity Transformation ¿¿¿¿¿ TBD

1.6 Linear Systems of Equations

The notions of rank and determinant not only help with determining whether a set of vectors are independent or not, they also help with solving linear systems of equations.

Consider a general linear system of equation

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_1\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

containing m equations and n unknowns $x_1, \dots, x_n$. Now you can easily see that this system can be written in matrix form as

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1.10}$$

So the question is: for a given $\mathbf{A}$ and $\mathbf{b}$, find all $\mathbf{x}$ that solve this equation. In general, Eq. (1.10) can have 0, 1, or infinitely many solutions.

The above question is essentially composed of two questions:

1) Does there exist any solutions to Eq. (1.10)?
2) If any solutions exist, is it unique?

We answer them in order:

1.6.1 Existence of solutions & range

Recall from the end of Section 1.1 that $\mathbf{A}$ defines a map from $\mathbb{F}^n$ to $\mathbb{F}^m$. Now, we are given a specific vector $\mathbf{b}$ in $\mathbb{F}^m$ and asked whether there are any vectors $\mathbf{x}$ in $\mathbb{F}^n$ that map to it. Again, the property in Eq. (1.2) comes in handy. Notice that

$$\mathbf{A}\mathbf{x} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

and so

$$\mathbf{A}\mathbf{x} \in \langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \rangle$$

This vector space, $\langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \rangle$, is so important that is given a name:

Definition 1.6.2 (Range of matrix) The "range space" or "column space" of a matrix $\mathbf{A}$, shown by range($\mathbf{A}$), is the vector space V spanned by its columns. □

This kind of automatically answers the question of existence:

$$\underline{\text{The equation } \mathbf{A}\mathbf{x} = \mathbf{b} \text{ has at least one answer } \mathbf{x} \Leftrightarrow \mathbf{b} \in \text{range}(\mathbf{A})} \tag{1.11}$$

Example 1.6.3 (Existence of solution) Consider again Example 1.4.6. Let

$$\mathbf{A} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}, \qquad \mathbf{b} = \mathbf{v}_4.$$

To see if we $\mathbf{A}\mathbf{x} = \mathbf{b}$ has an answer, we first have to build the range space of $\mathbf{A}$. As we saw in that example,

$$V = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$$

and so the range space of $\mathbf{A}$ is the same vector space V shown therein. But notice that the extra column $\mathbf{v}_3$ in $\mathbf{A}$ has no effect on its range space, because it is linearly dependent on the first two columns.

Does $\mathbf{v}_4$ belong to this vector space? Indeed, as we also showed in that example. So, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has at least one solution, for example, $\mathbf{x} = \begin{bmatrix}1 & -1 & 0\end{bmatrix}^T$.

In contrast, the equation

$$\mathbf{A}\mathbf{x} = \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

does not have any solutions, because $\mathbf{c} \notin V$. How do we check that $\mathbf{c} \notin V$? Well, if $\mathbf{c} \in V$, then $\mathbf{c}$ would be in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$, and so the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{c} \end{bmatrix}$$

would *not* be full rank. But this matrix is indeed full rank (check its determinant), which means that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{c}$ are linearly independent, and so $\mathbf{c} \notin V$. □

What I did in the example above is a general way of checking if $\mathbf{b} \in \text{range}(\mathbf{A})$: in order to check if $\mathbf{b}$ is linearly dependent on the columns of $\mathbf{A}$, you can just create a larger augmented matrix

$$\mathbf{A}_{\text{aug}} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$$

where you append $\mathbf{b}$ to the columns of $\mathbf{A}$. Then

$$\begin{aligned} &\text{if } \text{rank}(\mathbf{A}_{\text{aug}}) = \text{rank}(\mathbf{A}) \Rightarrow \mathbf{b} \in \text{range}(\mathbf{A}) \\ &\text{if } \text{rank}(\mathbf{A}_{\text{aug}}) > \text{rank}(\mathbf{A}) \Rightarrow \mathbf{b} \notin \text{range}(\mathbf{A}) \end{aligned}$$

and you can check the ranks either using determinants via Theorem 1.4.11, or directly using `rank()` in MATLAB. Note that if you are using the determinants, your job often becomes easier if you remove any linearly dependent columns of $\mathbf{A}$ before appending $\mathbf{b}$ to it, as we did in the example above.

1.6.4 Uniqueness of Solutions

Let us continue with the example above.

Example 1.6.5 (Uniqueness of solutions) Consider the same $\mathbf{A}$ and $\mathbf{b}$ as above,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

I gave one solution $\mathbf{x}$ above, but that is not the only solution. $\mathbf{x}' = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}^T$ also solves this equation, so does $\mathbf{x}'' = \begin{bmatrix} 3 & 1 & -2 \end{bmatrix}^T$, and many others. Why is this? Because

$$\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0 \Rightarrow \mathbf{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{z}} = 0$$

And so, for any scalar α,

$$\mathbf{A}(\mathbf{x} + \alpha\mathbf{z}) = \mathbf{A}\mathbf{x} + \alpha\mathbf{A}\mathbf{z} = \mathbf{b} + 0 = \mathbf{b}$$

so all $\mathbf{x} + \alpha\mathbf{z}$ are solutions as well. Note that this only happens because there exists a nonzero vector $\mathbf{z}$ such that $\mathbf{A}\mathbf{z} = 0$. □

The above examples motivates the definition of another fundamental concept in linear algebra:

Definition 1.6.6 (Null space) The null space of a matrix $\mathbf{A}$ is the vector space

$$\{\mathbf{z} \mid \mathbf{A}\mathbf{z} = 0\}.$$

□

Let me emphasize that the null space of a matrix is a vector space (can you show it?). In particular, it is never empty because it always contains at least the zero vector.

So to determine whether the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ (assuming that at least one exists) is unique, we need to determine whether the null space of $\mathbf{A}$ contains any nonzero vectors. Again, Eq. (1.2) comes in handy!

$$\mathbf{A}\mathbf{z} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{bmatrix}\begin{bmatrix}z_1 \\ z_2 \\ \vdots \\ z_n\end{bmatrix} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \cdots + z_n\mathbf{a}_n$$

and so the question of whether

$$\mathbf{A}\mathbf{z} = 0$$

for a nonzero $\mathbf{z}$ is precisely the question of whether

$$z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \cdots + z_n\mathbf{a}_n = 0$$

for a nonzero set of coefficients $z_1, z_2, \ldots, z_n$, which is precisely the definition of linear independence! So,

$$\underline{\text{The solution to } \mathbf{A}\mathbf{x} = \mathbf{b} \text{ (if any) is unique} \Leftrightarrow \mathbf{A} \text{ has full column rank}} \tag{1.12}$$

When the solution to the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ is not unique, then you essentially have a *solution space* rather than a solution. Obtaining the solution space is very easy once you have the null space: if $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k\}$ is a basis for the null space of $\mathbf{A}$ and $\mathbf{x}$ is one solution to the equation, then

$$\text{Solution space} = \{\mathbf{x} + \alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \cdots + \alpha_k\mathbf{z}_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{F}\}. \tag{1.13}$$

1.6.7 Finding the Solutions

At this point you might ask: ok, even if a unique solution exists, how to find it? This is done using the notion of matrix inverse:

Definition 1.6.8 (Matrix inverse) For a square and nonsignular matrix $\mathbf{A}$, there exists a unique matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix

$$\mathbf{I} = \begin{bmatrix}1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1\end{bmatrix}.$$

□

You may remember that for a 2 by 2 matrix, its inverse is given by

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For larger matrices, we will use MATLAB to find their inverses.

Let us now see how we can use matrix inverse to solve systems of linear equations:

- If $m = n$ ($\mathbf{A}$ is square) and $\mathbf{A}$ is nonsingular, then its columns span the entire $\mathbb{F}^n$ and therefore any $\mathbf{b}$ belongs to range($\mathbf{A}$) $= \mathbb{F}^n$. Therefore, from Eq. (1.11), at least one solution exists. But this solution is also unique from Eq. (1.12), because $\mathbf{A}$ has full column rank as well. To find this unique solution, simply multiply both sides of $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^{-1}$ *from the left*:
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{1.14}$$
- If $m > n$ ($\mathbf{A}$ is a tall matrix), and $\mathbf{A}$ is full column rank (it cannot be full row-rank, right?), then we have a unique solution only if $\mathbf{b} \in \text{range}(\mathbf{A}) \neq \mathbb{F}^m$. If so, to find that solution,
$$\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b} \Rightarrow (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Ax} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \Rightarrow \mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$
- If $m < n$ ($\mathbf{A}$ is a fat matrix), the equation never has a unique solution (even if $\mathbf{A}$ is full row-rank), because $\mathbf{A}$ cannot be full column rank (right?). Given any solution (that you can find, e.g., using elimination of variables), you can build the entire solution space as in Eq. (1.13).
  If, however, $\mathbf{A}$ is full row rank, then we are sure that the system of equations has always a solution for any $\mathbf{b}$ (why?). In this case, we also know (from a theorem we don't prove) that the square matrix $\mathbf{AA}^T$ is nonsingular, and
$$\mathbf{x} = \mathbf{A}^T(\mathbf{AA}^T)^{-1}\mathbf{b}$$
  is one solution to the equation (just plug it in and check!).

If you compare the three case above, you will notice that the matrices $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ (for full column rank $\mathbf{A}$) and $\mathbf{A}^T(\mathbf{AA}^T)^{-1}$ (for full row rank $\mathbf{A}$) are essentially taking the place of $\mathbf{A}^{-1}$ in Eq. (1.14). In fact, if $\mathbf{A}$ is square and nonsingular, both of them will become equal to $\mathbf{A}^{-1}$ (because $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$). In other words, $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ is an extension of $\mathbf{A}^{-1}$ for non-square, full column rank matrices and $\mathbf{A}^T(\mathbf{AA}^T)^{-1}$ is an extension of $\mathbf{A}^{-1}$ for non-square, full row rank matrices. As such, both of them are called the "pseudo-inverse" of $\mathbf{A}$, shown as $\mathbf{A}^\dagger$. Therefore, whenever $\mathbf{A}$ has full rank,

$$\mathbf{x} = \mathbf{A}^\dagger\mathbf{b} \tag{1.15}$$

is a solution to $\mathbf{Ax} = \mathbf{b}$. It goes beyond our course, but be aware that $\mathbf{A}^\dagger$ is always defined for any matrix (even the zero matrix) and Eq. (1.15) is always a solution to $\mathbf{Ax} = \mathbf{b}$.

MATLAB 1.6.9 (Inverse and pseudo-inverse) To find the inverse of a matrix, use the `inv()` function. Similarly, use `pinv()` to find the pseudo-inverse. However, if you want to invert a matrix only for the purpose of solving a linear equation, as in Eq. (1.14) or Eq. (1.15), a computationally advantageous way is to use the MATLAB's left division:

```
x = inv(A) * b; % Using matrix inverse
x = pinv(A) * b; % Using matrix pseudo-inverse
x = A \ b; % Using left division
```

Using left division also allows you to solve systems of equations without a unique solution, or even non-square systems of equations. If your system of equations has infinitely many solutions, `A \ b` returns one of them. If your system has no solutions, then it returns an $\mathbf{x}$ for which the error $\mathbf{Ax} - \mathbf{b}$ is smallest (in magnitude).□

1.7 Eigenvalues and Eigenvectors

OK, you have made it so far, and we are finally ready to learn about eigenvectors and eigenvalues! You'll see later why I put so much emphasis on them – they are one of the most important and widely used constructs in control theory.

Throughout this section, I will focus on square matrices $\mathbf{A} \in \mathbb{F}^{n \times n}$, because eigenvalues and eigenvectors are only defined for square matrices.

Recall, from Section 1.1, that matrices are not just arrays of numbers, but mappings from one vector space to another. So $\mathbf{A}$ maps from $\mathbb{F}^n$ to $\mathbb{F}^n$. In some cases, we can very easily describe what this mapping does:

Example 1.7.1 (2D mappings) Consider a few simple mappings in two dimensions:

- $\mathbf{A} = \mathbf{I}$ maps any vector to itself (identity mapping)
- $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects any vector with respect to the vertical axis

$$\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

  Similarly, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects any vector with respect to the horizontal axis.

- $\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k\mathbf{I}$ for $k > 0$ scales any vector by a factor of k

$$\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

- $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$ reflects any vector with respect to the origin

$$\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

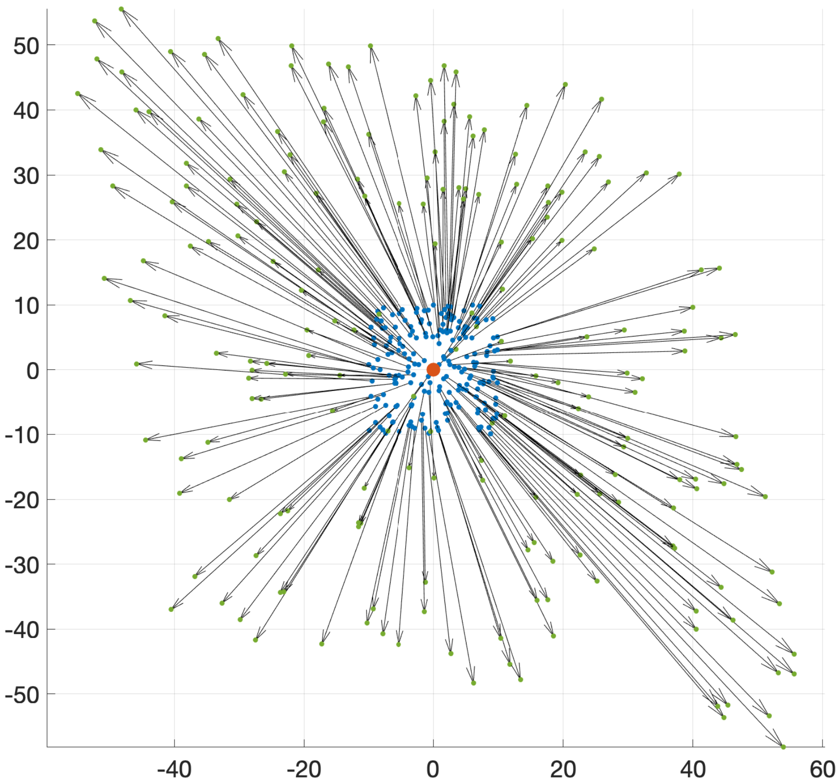
- $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates any vector as much as θ radians counter-clockwise.

□

What about more complex matrices? For example, how can we describe (or even intuitively understand) what does

$$\mathbf{A} = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \tag{1.16}$$

do to vectors? Here is how it maps a whole bunch of random points (the red dot shows the origin, the blue dots are random $\mathbf{x}$'s, the red dots are the corresponding $\mathbf{Ax}$, and the arrow shows the mapping):



It already gives us a sense: The arrows are all pointing outwards, showing that $\mathbf{A}$ perform some sort of enlargement (scaling with a scale $k > 1$). But this enlargement is not uniform, it is more pronounced along a NorthWest-SouthEast axis. Notice that this NorthWest-SouthEast axis can be described by the vector

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This vector is indeed special, since

$$\mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = 6\mathbf{v}_1$$

In other words, $\mathbf{v}_1$ is special because when $\mathbf{A}$ acts on it, the result is a multiple of $\mathbf{v}_1$ again! In other words, the effect of $\mathbf{A}$ on $\mathbf{v}_1$ is a pure scaling. (If you think this is not so special, try a whole bunch of random vectors and check if $\mathbf{A}\mathbf{v}$ becomes exactly a multiple of $\mathbf{v}$.)

Note that this special property also clearly holds for any multiple of $\mathbf{v}_1$:

$$\mathbf{A}(\alpha\mathbf{v}_1) = \alpha\mathbf{A}\mathbf{v}_1 = \alpha 6\mathbf{v}_1 = 6(\alpha\mathbf{v}_1)$$

In other words, the effect of $\mathbf{A}$ on the whole NorthWest-SouthEast axis is a 6-time enlargement.

But this does not tell us all about $\mathbf{A}$. What about other directions other than the NorthWest-SouthEast axis? We can visually see that no other direction is scaled as much. To see what $\mathbf{A}$ does to other vectors, we can search to see if there are any other vectors such that the effect of $\mathbf{A}$ on them is a pure scaling. In other words, are there any vectors $\mathbf{v}$, other than $\mathbf{v}_1$ and its multiples, such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{1.17}$$

for some scalar λ. Clearly, $\mathbf{v} = 0$ satisfies this, but that is not what we are looking for.

Eq. (1.17) is fortunately a linear system of equations, equivalently written as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$$

and we are looking for nonzero vectors $\mathbf{v}$ that satisfy it. The difficulty with respect to a usual linear system of equations is that λ is also unknown. But notice one thing. If λ is such that $\mathbf{A} - \lambda\mathbf{I}$ is nonsingular, then

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0 \Rightarrow (\mathbf{A} - \lambda\mathbf{I})^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})^{-1}0 \Rightarrow \mathbf{v} = 0$$

In other words, for any λ such that $\mathbf{A} - \lambda\mathbf{I}$ is nonsingular, Eq. (1.17) has only the unique solution $\mathbf{v} = 0$, which is of no help. This is very useful, because we know we have to restrict our attention to values of λ for which $\mathbf{A} - \lambda\mathbf{I}$ is singular:

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) = 0 &\Leftrightarrow \det\left(\begin{bmatrix}5-\lambda & -1\\ -1 & 5-\lambda\end{bmatrix}\right) = 0\\ &\Leftrightarrow (5-\lambda)^2 - 1 = 0\\ &\Leftrightarrow \lambda^2 - 10\lambda + 24 = 0\end{aligned}$$

This already gives a polynomial equation in λ, with solutions

$$\lambda_1 = 6$$
$$\lambda_2 = 4$$

$\lambda_1 = 6$ is what we had originally found by guessing $\mathbf{v}_1$. So for λ_1, there is even no need to solve Eq. (1.17), because we already know its solution. But what about the solution to $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v} = 0$?

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v} = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}v_1 - v_2\\ v_2 - v_1\end{bmatrix} \tag{1.18}$$

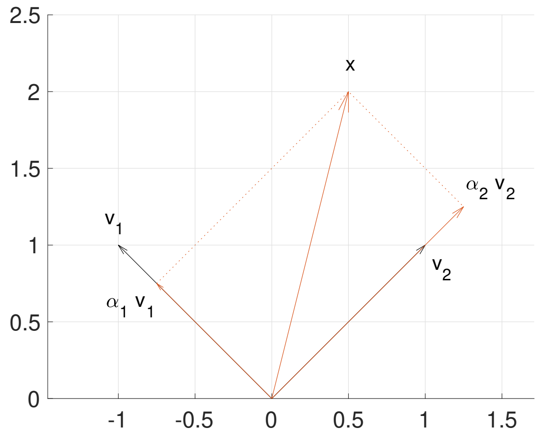
So $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v} = 0$ if and only if $v_1 = v_2$. Not surprisingly, we did not get a unique solution $\mathbf{v}$, because we found λ_2 precisely such that we get infinitely many solutions. It is not hard to see that vectors $\mathbf{v}$ for which $v_1 = v_2$ are all multiples of

$$\mathbf{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$$

and constitute the SouthWest-NorthEast axis in the picture. Now the picture makes even more sense: the mapping $\mathbf{A}$ scales all the vectors along the NorthWest-SouthEast axis by 6 times, and all the vectors along the SouthWest-NorthEast axis by 4 times.

What about other vectors $\mathbf{x}$ that lie neither on the NorthWest-SouthEast axis nor on the SouthWest-NorthEast axis? Well, notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, and therefore create a basis for $\mathbb{R}^2$. So, all we need to do is change our basis from the standard basis $\mathbf{I} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}$ to the new basis $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}$, which we know how to do from Section 1.5:

$$\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$$
$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \mathbf{V}^{-1}\mathbf{x}$$



Then, we can clearly see what happens when $\mathbf{A}$ is applied to $\mathbf{x}$:

$$\begin{aligned}\mathbf{Ax} &= \mathbf{A}(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) \\ &= \alpha_1\mathbf{Av}_1 + \alpha_2\mathbf{Av}_2 \\ &= 6\alpha_1\mathbf{v}_1 + 4\alpha_2\mathbf{v}_2\end{aligned}$$

So $\mathbf{A}$ scales the component of any vector along the NorthWest-SouthEast axis (along $\mathbf{v}_1$) by 6 times and its component along the SouthWest-NorthEast axis (along $\mathbf{v}_2$) by 4 times. That's it, as simple as it will ever get!

Definition 1.7.2 (Eigenvalue and eigenvector) For any matrix $\mathbf{A} \in \mathbb{F}^{n\times n}$, there exist precisely n numbers λ (potentially complex, and potentially repeated) for which the equation

$$\mathbf{Av} = \lambda\mathbf{v}$$

has nonzero solutions. These numbers are called the eigenvalues of $\mathbf{A}$, and the corresponding nonzero vectors $\mathbf{v}$ that solve this equation are called the eigenvectors of $\mathbf{A}$. □

Let's see a few examples of finding eigenvalues and eigenvectors.

Example 1.7.3 (3x3 matrix with unique eigenvalues) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 3 & 7 \\ -5 & 3 & 5 \\ -4 & 2 & 6 \end{bmatrix}$$

To find its eigenvalues, we need to solve the equation

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\ \begin{vmatrix} -5-\lambda & 3 & 7 \\ -5 & 3-\lambda & 5 \\ -4 & 2 & 6-\lambda \end{vmatrix} &= 0 \\ -\lambda^3 + 4\lambda^2 - 6\lambda + 4 &= 0\end{aligned}$$

This a polynomial equation and we know, from Section **??**, that it has exactly 3 (potentially repeated, potentially complex) roots. Using hand calculations or MATLAB, we can see that its solutions are

$$\lambda_1 = 2$$
$$\lambda_2 = 1 + j$$
$$\lambda_3 = 1 - j$$

which we sometimes show more compactly as

$$\lambda_1 = 2$$
$$\lambda_{2,3} = 1 \pm j$$

To find the eigenvectors associated with each eigenvalues, we simply solve the equation $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v} = 0$, as we did in Eq. (1.18). Note that this is nothing but finding the null space of $\mathbf{A} - \lambda_i \mathbf{I}$. For λ_1, this becomes

$$(\mathbf{A} - 2\mathbf{I}) \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{v}} = 0 \Leftrightarrow \begin{cases} -7a + 3b + 7c = 0 \\ -5a + b + 5c = 0 \Leftrightarrow b = 5a - 5c \\ -4a + 2b + 4c = 0 \end{cases}$$

Substituting $b = 5a - 5c$ into the other two equations then gives

$$\begin{cases} 8a - 8c = 0 \\ 6a - 6c = 0 \end{cases} \Leftrightarrow c = a$$

Therefore, any vector

$$\mathbf{v}_1 = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix}, \qquad \text{for any } a \in \mathbb{C}, a \neq 0$$

is an eigenvector corresponding to $\lambda_1 = 2$. This gives an entire line in the 3D space that is scaled by 2 by $\mathbf{A}$ (similar to the NorthWest-SouthEast and SouthWest-NorthEast directions for the matrix $\mathbf{A}$ in Eq. (1.16)). If you prefer (or are asked to provide) a single eigenvector associated with λ_1, pick any $a \in \mathbb{C}$ that you like, for example $\mathbf{v}_1 = \begin{bmatrix} 2j \\ 0 \\ 2j \end{bmatrix}$.

To find the eigenvector associated with λ_2, we proceed similarly:

$$(\mathbf{A} - (1+j)\mathbf{I}) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Leftrightarrow \begin{cases} -(6+j)a + 3b + 7c = 0 \\ -5a + (2-j)b + 5c = 0 \\ -4a + 2b + (5-j)c = 0 \Leftrightarrow b = 2a - \frac{5-j}{2}c \end{cases}$$

Substituting the last equation into the first two gives

$$\begin{cases} -ja - (0.5 - 1.5j)c = 0 \\ -(1+2j)a + (0.5 + 3.5j)c = 0 \end{cases}$$

These equations may not immediately look multiples of each other, but they are in fact. To see, solve one and replace in the other:

$$\text{1st eq } \Leftrightarrow a = (1.5 + 0.5j)c \overset{\text{2nd eq}}{\Rightarrow} -(0.5 + 3.5j)c + (0.5 + 3.5j)c = 0$$

which holds for any c. Substituting $a = (1.5 + 0.5j)c$ into $b = 2a - \frac{5-j}{2}c$ also gives us $b = (0.5 + 1.5j)c$. Therefore, any vector

$$\mathbf{v}_2 = \begin{bmatrix} (1.5 + 0.5j)c \\ (0.5 + 1.5j)c \\ c \end{bmatrix}, \qquad \text{for any } c \in \mathbb{C}, c \neq 0$$

is an eigenvector associated with $\lambda_2 = 1 + j$. Again, if you want a single matrix, pick your choice of $c \neq 0$, such as $c = 2 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 3+j & 1+3j & 2 \end{bmatrix}^T$.

Finally, to find the eigenvector associated with $\lambda_3 = 1 - j$, you can repeat the same process as above, which gives,

$$\mathbf{v}_3 = \begin{bmatrix} (1.5 - 0.5j)c \\ (0.5 - 1.5j)c \\ c \end{bmatrix}, \qquad \text{for any } c \in \mathbb{C}, c \neq 0$$

□

Notice that $\mathbf{v}_3$ is the complex conjugate of $\mathbf{v}_2$. This is not by chance. Whenever you have two eigenvalues that are complex conjugates of each other, their corresponding eigenvectors are also complex conjugate of each other (can you prove this?).

In the examples that we have seen so far, all the eigenvalues of $\mathbf{A}$ have been distinct. This is not always the case. The following is an example.

Example 1.7.4 (3x3 matrix with repeated eigenvalues) This time consider the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 & -6 \\ 6 & -1 & 6 \\ 6 & -2 & 7 \end{bmatrix}$$

Similar to previous examples, eigenvalue are found using

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$
$$\begin{vmatrix} -5-\lambda & 2 & -6 \\ 6 & -1-\lambda & 6 \\ 6 & -2 & 7-\lambda \end{vmatrix} = 0$$
$$-\lambda^3 + \lambda^2 + \lambda - 1 = 0$$

This equation has three roots equal to

$$\lambda_1 = -1$$
$$\lambda_2 = \lambda_3 = 1$$

two of which are repeated. This is perfectly fine. However, finding the eigenvectors becomes a bit more complicated. For $\lambda_1 = -1$ which is *not* repeated, everything is as before. You solve $(\mathbf{A} + \mathbf{I})\mathbf{v} = 0$ and will find that any vector

$$\mathbf{v}_1 = \begin{bmatrix} a \\ -a \\ -a \end{bmatrix}, \qquad \text{for any } a \in \mathbb{C}, a \neq 0$$

is an eigenvector corresponding to λ_1.

In order to find the eigenvectors corresponding to both of λ_2 and λ_3, we have to solve the same equation

$$(\mathbf{A} - \mathbf{I}) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Leftrightarrow \begin{cases} -6a + 2b - 6c = 0 \\ -6a + 2b - 6c = 0 \\ -6a + 2b - 6c = 0 \end{cases} \Leftrightarrow b = 3a + 3c$$

You can see that this time, we get only one equation in 3 variables, and we have two free variables to choose arbitrarily (I chose a and c, but any two would work). Therefore, any vector

$$\mathbf{v}_{2,3} = \begin{bmatrix} a \\ 3a + 3c \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \qquad \text{for any } a, c \in \mathbb{C}, (a, c) \neq 0 \tag{1.19}$$

is an eigenvector corresponding to $\lambda_{2,3} = 1$. Notice the difference with the example before where the eigenvalues where different. When the eigenvalues where different, we found one <u>line</u> of eigenvectors corresponding to each eigenvalue. Now that we have a repeated eigenvalue, we found a <u>plane</u> of eigenvectors, with the same dimension (2) as the multiplicity of the repeated eigenvalue.

Similar to before, if you want (or are asked to) give <u>two specific eigenvectors</u> corresponding to λ_2 and λ_3 (instead of the plane of eigenvectors in Eq. (1.19)), you can pick any two <u>linearly independent</u> vectors from that plane, for example,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

□

The case of repeated eigenvalues can get more complex than this. In the above example, we were able to find two linearly independent vectors that satisfy $(\mathbf{A} - \mathbf{I})\mathbf{v} = 0$. In other words, the dimension of the null space of $\mathbf{A} - \mathbf{I}$ was 2, equal to the multiplicity of the repeated eigenvalues. We may not always be so lucky! Consider for example the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

It is not hard to see that the two eigenvalues are $\lambda_1 = \lambda_2 = 3$. So we ideally would need two linearly independent vectors that satisfy $(\mathbf{A} - 3\mathbf{I})\mathbf{v} = 0$. This is impossible, because the null space of $\mathbf{A} - 3\mathbf{I}$ is only 1 dimensional. To see this, notice that

$$(\mathbf{A} - 3\mathbf{I}) \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Leftrightarrow \begin{cases} b = 0 \\ 0 = 0 \end{cases}$$

So we can only choose a freely, but b must be zero, and only the vectors

$$\mathbf{v}_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}, \qquad \text{for any } a \in \mathbb{C}, a \neq 0$$

are an eigenvector corresponding to $\lambda_1 = \lambda_2 = 3$. What about the other eigenvector $\mathbf{v}_2$? It doesn't exist! For these kinds of matrices where not enough eigenvectors can be found (which can *only* happen if we have repeated eigenvalues), we have to supplement the eigenvectors with additional vectors called "generalized eigenvectors". Good news, that's beyond our course!

Let's go back to Example 1.7.4 where we were lucky and able to find two linearly independent eigenvectors corresponding to the repeated eigenvalue. You might have noticed that not only $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, but all three eigenvalues $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent. The same was true in Example 1.7.3 where the eigenvalues were distinct. This is again not by chance:

Theorem 1.7.5 (Independence of eigenvectors) Consider a matrix $\mathbf{A} \in \mathbb{F}^{n\times n}$ that has distinct eigenvalues, or a matrix $\mathbf{A} \in \mathbb{F}^{n\times n}$ that has repeated eigenvalues but we are able to find as many linearly independent eigenvectors as the number of repeated eigenvalues. In both cases, the set of all eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent. □

In the above theorem, I focused on a specific set of (square) matrices: those that either have distinct eigenvalues, or even if they have repeated eigenvalues, we are able to find as many independent eigenvectors as the multiplicity of each repeated eigenvalue. For reasons that we will see shortly, these matrices are called diagonalizable.

Before closing this section, here is how to do all of these in MATLAB.

MATLAB 1.7.6 (Eigenvalues & eigenvectors) The function `eig()` gives you the eigenvalues and eigenvectors. If you only want the eigenvalues, type

```
lambda = eig(A);
```

and it will give you a column vector `lambda` containing the eigenvalues of `A`. If you want the eigenvectors as well, type

```
[V, D] = eig(A);
```

and it will give you two matrices the same size as `A`. `D` is a diagonal matrix with the eigenvalues of `A` on its diagonal, and `V` is a matrix with eigenvectors of `A` as its columns (with the correct ordering, such that the i'th column of `V` is the eigenvector corresponding to the i'th diagonal element of `D`. □

1.7.7 Diagonalization

The process of diagonalization is one in which a square matrix is "transformed" into a diagonal one using a change of basis. Here is how. If you are not fresh on Section 1.5 (Change of Basis), it's a good time to review it!

Consider a *diagonalizable* matrix $\mathbf{A} \in \mathbb{F}^{n\times n}$ (which you know what it means from the last section, right?). As always, we look at $\mathbf{A}$ as a map from $\mathbb{F}^n$ to $\mathbb{F}^n$:

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{1.20}$$

Now let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$ (potentially repeated) and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the corresponding linearly independent eigenvectors. The matrix

$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}$$

is nonsingular, and we can use if for a change of basis. The vectors $\mathbf{x}$ and $\mathbf{y}$ are in the standard basis

$$\mathbf{I} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{bmatrix}$$

and we move them both to the new basis given by $\mathbf{V}$ (again, if these are not making full sense, make sure you review Section 1.5). The new representations of them in the new basis is given by

$$\hat{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x}, \qquad \hat{\mathbf{y}} = \mathbf{V}^{-1}\mathbf{y}$$

Substituting these into Eq. (1.20), we get

$$\begin{aligned}\mathbf{V}\hat{\mathbf{y}} &= \mathbf{A}\mathbf{V}\hat{\mathbf{x}}\\ \hat{\mathbf{y}} &= \underbrace{\mathbf{V}^{-1}\mathbf{A}\mathbf{V}}_{\hat{\mathbf{A}}}\hat{\mathbf{x}}\\ \hat{\mathbf{y}} &= \hat{\mathbf{A}}\hat{\mathbf{x}}\end{aligned}$$

So the matrix $\hat{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is the representation of the same linear map as $\mathbf{A}$, but in the new basis $\mathbf{V}$.

The story gets more interesting. Let's look at $\hat{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ more closely. First, let us look at the product

$$\mathbf{A}\mathbf{V} = \mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}$$

Recall from Eq. (1.1) that this is equal to

$$\mathbf{A}\mathbf{V} = \begin{bmatrix}\mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n\end{bmatrix}$$

but $\mathbf{v}_i$ are not any vectors, they are the eigenvectors of $\mathbf{A}$, so this simplifies to

$$\begin{aligned}\mathbf{A}\mathbf{V} &= \begin{bmatrix}\lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}\end{aligned}$$

Can you convince yourself of the last equality? Notice that this is nothing but the product of two matrices, so you can first break it into n matrix-vector products from Eq. (1.1), each of which is a linear combination from Eq. (1.2). The matrix

$$\mathbf{\Lambda} = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}$$

is an important matrix for us, because it contains the eigenvalues of $\mathbf{A}$ on its diagonal. Using this new notation, we get

$$\mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}$$

which immensely simplifies $\hat{\mathbf{A}}$:

$$\hat{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda} = \mathbf{\Lambda}$$

In other words, the representation of a matrix $\mathbf{A}$ in the basis of its eigenvectors is nothing but the diagonal matrix of its eigenvalues! This is called "diagonalization", and $\mathbf{\Lambda}$ is also called the diagonalization/diagonalized version of $\mathbf{A}$.

If these seem like a lot to digest, notice that this is what we did at the beginning of Section 1.7 where we were analyzing what the matrix $\begin{bmatrix}5 & -1\\ -1 & 5\end{bmatrix}$ does to vectors in the plane. After going step by step through the extraction of eigenvalues and eigenvectors, at the end we decomposed any vector $\mathbf{x}$ into its components along $\mathbf{v}_1$ and $\mathbf{v}_2$ and used the fact that in this new "coordinate system", $\mathbf{A}$ becomes a pure scaling in each direction, which is precisely what the diagonal matrix $\mathbf{\Lambda}$ does. Diagonalization will be a very valuable tool later in the study of linear systems.

1.8 Singular Value Decomposition

Consider a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and assume it is diagonalizable. Then, as we saw above,

$$\mathbf{AV} = \mathbf{V\Lambda}$$

Note that this can be re-written as

$$\mathbf{V}^{-1}\mathbf{A} = \mathbf{\Lambda V}^{-1}$$

which may not look useful, except if you let

$$\mathbf{W} = \mathbf{V}^{-*}$$

so

$$\mathbf{W}^*\mathbf{A} = \mathbf{\Lambda W}^*$$

This simply means that $\mathbf{W}$ is the matrix of *left eigenvectors* of $\mathbf{A}$ corresponding to the same eigenvalues. Notice that by convention, we have chosen $\mathbf{W}$ such that its *columns* are the left eigenvectors, even though the transpose of its columns (which become rows) multiply $\mathbf{A}$ from the left. Now you can take this further and see that

$$\mathbf{A} = \mathbf{V\Lambda W}^* = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}\begin{bmatrix}\mathbf{w}_1^*\\ \vdots\\ \mathbf{w}_n^*\end{bmatrix} = \sum_{i=1}^n \lambda_i \mathbf{v}_i \mathbf{w}_i^* \tag{1.21}$$

Exercise 1.8.1 (Rank of vector outer product) If $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ are two vectors, what is the rank of the matrix $\mathbf{vw}^*$? □

In other words, $\mathbf{A}$ can be *decomposed* into the sum of n, rank-1 matrices $\lambda_i \mathbf{v}_i \mathbf{w}_i^*$ using its eigen-decomposition. This is extremely useful for approximating $\mathbf{A}$ in situations when $\mathbf{A}$ is large-dimensional but has only a few non-zero (or non-negligible) eigenvalues. However, the rank-1 decomposition in Eq. (1.21) has two problems:

- It only works for square matrices
- It even only works for diagonalizable square matrices
- More importantly, the eigenvalue/vectors can be complex even for real matrices
- Even more importantly, computing the eigen-decomposition is a numerically challenging and unstable problem

The singular value decomposition (SVD) resolves all of these problems.

Definition 1.8.2 (Singular values) For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, let $r = \min\{m, n\}$. The square root of the largest r eigenvalues of $\mathbf{A}^T\mathbf{A}$ are called the singular values of $\mathbf{A}$. □

Definition 1.8.2 may sound complicated at first, but it will become very straightforward when we notice a few facts:

- The matrices $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ are both square and symmetric but have different sizes if $m \neq n$.
- The matrices $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ are both positive semidefinite (to be defined shortly in the next section) and therefore all of their eigenvalues are real and nonnegative.

- The eigenvalues of whichever of $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ that has a larger dimension consists of the r eigenvalues of the other (smaller) one and $\max\{m,n\}-r$ zeros. In other words, modulo some additional zeros, the eigenvalues of $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are the same.

So in order to get the singular values of $\mathbf{A}$, we can pick either of $\mathbf{A}^T\mathbf{A}$ or $\mathbf{A}\mathbf{A}^T$, compute its eigenvalues, and sort them in descending order

$$\sigma_1^2 \ge \sigma_2^2 \ge \cdots \ge \sigma_r^2 \ge \underbrace{0 = \cdots = 0}_{\text{if using the larger of } \mathbf{A}^T\mathbf{A} \text{ and } \mathbf{A}\mathbf{A}^T}$$

The singular values of $\mathbf{A}$ are then

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r$$

Notice that some or even all of the singular values can still be zero, depending on the rank of $\mathbf{A}$. More precisely, exactly $\text{rank}(\mathbf{A})$ of the singular values of $\mathbf{A}$ are positive and the remaining ones are zero.

Exercise 1.8.3 (Rank and nonzero singular values) Prove the above relationship between rank and nonzero singular values. You can prove and use an intermediate fact that $\text{rank}(\mathbf{A}\mathbf{A}^T) = \text{rank}(\mathbf{A}^T\mathbf{A}) = \text{rank}(\mathbf{A})$. □

Now that we know what singular values are, we can define the SVD:

Theorem 1.8.4 (SVD) For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, there exists a unique decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.22}$$

where $\mathbf{U} \in \mathbb{R}^{m\times m}$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$ are orthogonal matrices and $\mathbf{S} \in \mathbb{R}^{m\times n}$ is a diagonal matrix with the singular values of $\mathbf{A}$ on its diagonal. □

Recall that our original motivation for going after the SVD was a sum of rank-1 decomposition similar to Eq. (1.21). This is precisely what Eq. (1.22) is giving us. To see how, assume for example that $\mathbf{A}$ is 5×3. Then Eq. (1.22) can be rewritten as

$$\mathbf{A} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 & \mathbf{u}_5 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{bmatrix}$$

You should be able to convince yourself from the rules of Theorem 1.1.3 that the red parts of $\mathbf{U}$ and $\mathbf{S}$ multiply each other, and so

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{bmatrix} \\ &= \sum_{i=1}^{3} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \end{aligned}$$

which is precisely a sum of rank-1 matrices as desired, with them already being ranked in the order of the significance of their contribution to the sum.

1.9 Symmetric Matrices: Definite, Semidefinite, or Indefinite

So far, our discussions about square matrices were quite general. At this last section, I want to emphasize how special and nice a lot of things become when we focus on symmetric matrices.

Recall that a symmetric matrix is a real matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{P}^T = \mathbf{P}$$

Symmetric matrices have a special property that:

Theorem 1.9.1 (Eigenvalues and eigenvectors) For any symmetric matrix, it holds that

(i) all of its eigenvalues are real;

(ii) it is diagonalizable (a set of n linearly independent eigenvectors always exists);

(iii) its eigenvectors are orthogonal to each other (which is stronger than linear independence).

□

These properties of symmetric matrices are particularly helpful when we study the so called "quadratic forms":

Definition 1.9.2 (Quadratic forms) Let $\mathbf{P}$ be a symmetric matrix. The matrix-to-scalar function

$$q(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$$

is called (to be in) a quadratic form. □

Quadratic forms come up a lot in linear systems and controls; in the study of stability, controllability, observability, and more. Whenever we have a quadratic form $\mathbf{x}^T\mathbf{P}\mathbf{x}$, five scenarios may occur (recall that the eigenvalues of $\mathbf{P}$ are all real):

1. All eigenvalues of $\mathbf{P}$ are positive. In this case, you can show (using the diagonalization of $\mathbf{P}$ and Theorem 1.9.1) that

   $$\mathbf{x}^T\mathbf{P}\mathbf{x} > 0 \qquad \text{for any } \mathbf{x} \neq 0$$

   and we call the matrix $\mathbf{P}$ "positive definite".

2. All eigenvalues of $\mathbf{P}$ are nonnegative (positive or zero). Similarly,

   $$\mathbf{x}^T\mathbf{P}\mathbf{x} \geq 0 \qquad \text{for any } \mathbf{x}$$

   and we call the matrix $\mathbf{P}$ "positive semidefinite".

3. All eigenvalues of $\mathbf{P}$ are negative. In this case,

   $$\mathbf{x}^T\mathbf{P}\mathbf{x} < 0 \qquad \text{for any } \mathbf{x} \neq 0$$

   and we call the matrix $\mathbf{P}$ "negative definite".

4. All eigenvalues of $\mathbf{P}$ are nonpositive (negative or zero). In this case,

   $$\mathbf{x}^T\mathbf{P}\mathbf{x} \leq 0 \qquad \text{for any } \mathbf{x}$$

   and we call the matrix $\mathbf{P}$ "negative semidefinite".

5. $\mathbf{P}$ has both positive and negative eigenvalues, in which case $\mathbf{x}^T\mathbf{P}\mathbf{x}$ may be both positive and negative, and we call $\mathbf{P}$ "indefinite".

To some degree, positive definite matrices are an extension of positive numbers to matrices, and similarly for the other categories. Finally, keep in mind that if you have a function $\mathbf{x}^T\mathbf{A}\mathbf{x}$ for a non-symmetric matrix $\mathbf{A}$, you can equivalently write it as the quadratic form

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T \underbrace{\left(\frac{\mathbf{A} + \mathbf{A}^T}{2}\right)}_{\mathbf{P}} \mathbf{x}$$

where $\mathbf{P} = \frac{\mathbf{A}+\mathbf{A}^T}{2}$ is called the *symmetric part* of $\mathbf{A}$ (right?).

References

[1] M. W. Spong, S. Hutchinson, and M. Vidyasagar, *Robot modeling and control.* John Wiley & Sons, 2020.